International **J**ournal of **P**ure and **A**pplied **M**athematics
Volume 39, No 4, pp. 475-488, 2007 ISSN 1311-8080

Various characterizations of Besov-Dunkl spaces

Chokri Abdelkefi§ and Feriel Sassi ‡1

§‡Department of Mathematics, Preparatory Institute of Engineer Studies of Tunis
1089 Monfleury Tunis, Tunisia
E-mail : chokri.abdelkefi@ipeit.rnu.tn
E-mail : feriel.sassi@fst.rnu.tn

Abstract: In this paper, different characterizations of the Besov-Dunkl spaces, previously considered in [1, 2, 3, 11], are given. We provide equivalence between these characterizations, using the Dunkl translation, the Dunkl transform and the Peetre K-functional.

AMS Subject Classification: 46E30, 44A15, 44A35.
Key Words : Dunkl operators, Dunkl transform, Dunkl translation operators, Dunkl convolution, Besov-Dunkl spaces .

1. Introduction

On the real line, we consider the first-order differential-difference operator defined by

$$\Lambda_\alpha(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha+1}{x}\left[\frac{f(x)-f(-x)}{2}\right], \quad f \in \mathcal{E}(\mathbb{R}), \quad \alpha > -\frac{1}{2},$$

which is called Dunkl operator. Such operators have been introduced in 1989, by C. Dunkl in [8]. The Dunkl kernel E_α is used to define the Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [9]. Rösler in [17] shows that the Dunkl kernel verify a product formula. This allows us to define the Dunkl translation τ_x , $x \in \mathbb{R}$. As a result, we have the Dunkl convolution.

There are many ways to define Besov spaces (see [4, 5, 15, 21]). This paper deals with Besov-Dunkl spaces (see [1, 2, 3, 11]). Let $1 \le p < +\infty$,

[1]Received: April 12, 2007

$1 \leq q \leq +\infty$ and $\beta > 0$, the Besov-Dunkl space denoted by $\mathcal{BD}_{p,q}^{\beta;\alpha}$ is the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying

$$\int_0^{+\infty} \left(\frac{w_{p,\alpha}(f,x)}{x^\beta} \right)^q \frac{dx}{x} \quad < \quad +\infty \qquad if \quad q < +\infty$$

and

$$\sup_{x \in (0,+\infty)} \frac{w_{p,\alpha}(f,x)}{x^\beta} \quad < \quad +\infty \qquad if \quad q = +\infty,$$

where $w_{p,\alpha}(f,x) = \sup_{|t| \leq x} \|\tau_t(f) - f\|_{p,\alpha}$ and μ_α is a weighted Lebesgue measure on $\mathbb{R}$ (see next section).
Put $\mathcal{D}_{p,\alpha}$ the subspace of functions $f \in L^p(\mu_\alpha)$ such that the distribution function $\Lambda_\alpha f \in L^p(\mu_\alpha)$. $\mathcal{D}_{p,\alpha}$ is a Banach space with $\|.\|_{\mathcal{D}_{p,\alpha}}$ defined by

$$\|f\|_{\mathcal{D}_{p,\alpha}} = \|f\|_{p,\alpha} + \|\Lambda_\alpha f\|_{p,\alpha}.$$

We consider the subspace $\mathcal{KD}_{p,q}^{\beta,\alpha}$ of functions $f \in L^p(\mu_\alpha)$ satisfying

$$\int_0^{+\infty} \left(\frac{K_{p,\alpha}(f,x)}{x^\beta} \right)^q \frac{dx}{x} \quad < \quad +\infty \qquad if \quad q < +\infty$$

and

$$\sup_{x \in (0,+\infty)} \frac{K_{p,\alpha}(f,x)}{x^\beta} \quad < \quad +\infty \qquad if \quad q = +\infty,$$

where $K_{p,\alpha}$ is the Peetre K-functional (see[12]) given by

$$K_{p,\alpha}(f,x) = \inf \Big\{ \|f_0\|_{p,\alpha} + x\|\Lambda_\alpha f_1\|_{p,\alpha}\,;\; f_0 \in L^p(\mu_\alpha),\; f_1 \in \mathcal{D}_{p,\alpha},\; f = f_0 + f_1 \Big\}.$$

We denote by $\mathcal{ED}_{p,q}^{\beta,\alpha}$ the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying

$$\int_1^{+\infty} \Big(x^\beta \mathbf{E}_{p,\alpha}(f,x) \Big)^q \frac{dx}{x} \quad < \quad +\infty \qquad if \quad q < +\infty$$

and

$$\sup_{x \in (1,+\infty)} x^\beta \mathbf{E}_{p,\alpha}(f,x) \quad < \quad +\infty \qquad if \quad q = +\infty,$$

where $\mathbf{E}_{p,\alpha}(f,x) = \inf \Big\{ \|f - g\|_{p,\alpha}\,;\; \operatorname{supp}(\mathcal{F}_\alpha(g)) \subset [-x,\, x] \Big\},\; x > 0.$

Our objective will be to prove that $\mathcal{BD}_{p,q}^{\beta,\alpha} = \mathcal{KD}_{p,q}^{\beta,\alpha}$ and when $1 \le p \le 2$, $1 \le q < +\infty$, $0 < \beta < 1$ then $\mathcal{BD}_{p,q}^{\beta,\alpha} = \mathcal{ED}_{p,q}^{\beta,\alpha}$.
Analogous results have been obtained by Betancor, Méndez and Rodríguez-Mesa in [6] for the Bessel operator on $(0, +\infty)$.

The contents of this paper are as follows.
In section 2, we collect some basic definitions and results about harmonic analysis associated with Dunkl operators .
In section 3, we prove the results about inclusion and coincidence between the spaces $\mathcal{BD}_{p,q}^{\beta,\alpha}$, $\mathcal{KD}_{p,q}^{\beta,\alpha}$ and $\mathcal{ED}_{p,q}^{\beta,\alpha}$.

In the sequel c represents a suitable positive constant which is not necessarily the same in each occurence. Furthermore, we denote by

- $\mathcal{D}_*(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$ with compact support.
- $\mathcal{S}_*(\mathbb{R})$ the space of even Schwartz functions on $\mathbb{R}$.

2. Preliminaries

Let μ_α the weighted Lebesgue measure on $\mathbb{R}$ given by

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dx.$$

For every $1 \le p \le +\infty$, we denote by $L^p(\mu_\alpha)$ the space $L^p(\mathbb{R}, d\mu_\alpha)$ and we use $\| \ \|_{p,\alpha}$ as a shorthand for $\| \ \|_{L^p(\mu_\alpha)}$.
The Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [9], is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} E_\alpha(-ixy)f(y)d\mu_\alpha(y), \quad x \in \mathbb{R},$$

where for $\lambda \in \mathbb{C}$, the Dunkl kernel $E_\alpha(\lambda.)$ is given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

with j_α the normalized Bessel function of the first kind and order α (see [22]).
The Dunkl kernel $E_\alpha(\lambda\ .)$ is the unique solution on $\mathbb{R}$ of initial problem for the Dunkl operator (see [8]). We have for all x, $y \in \mathbb{R}$,

$$|E_\alpha(-ixy)| \le 1. \tag{1}$$

According to [7], we have the following results :

i) For all $f \in L^1(\mu_\alpha)$, we have $\|\mathcal{F}_\alpha(f)\|_{\infty,\alpha} \leq \|f\|_{1,\alpha}$.

ii) For all $f \in L^1(\mu_\alpha)$ such that $\mathcal{F}_\alpha(f) \in L^1(\mu_\alpha)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}} E_\alpha(i\lambda x)\mathcal{F}_\alpha(f)(\lambda)d\mu_\alpha(\lambda),\ a.e\ x \in \mathbb{R}. \tag{2}$$

iii) For every $f \in L^2(\mu_\alpha)$, we have the Plancherel formula

$$\|\mathcal{F}_\alpha(f)\|_{2,\alpha} = \|f\|_{2,\alpha}.$$

For all $x, y, z \in \mathbb{R}$, consider

$$W_\alpha(x,y,z) = \frac{(\Gamma(\alpha+1)^2)}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}(1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x})\Delta_\alpha(x,y,z) \tag{3}$$

where

$$b_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, y \in \mathbb{R}\backslash\{0\},\ z \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_\alpha(x,y,z) = \begin{cases} \frac{([(|x|+|y|)^2-z^2][z^2-(|x|-|y|)^2])^{\alpha-\frac{1}{2}}}{|xyz|^{2\alpha}} & \text{if } |z| \in S_{x,y} \\ 0 & \text{otherwise} \end{cases}$$

where

$$S_{x,y} = \Big[||x| - |y|| \ ,\ |x| + |y| \Big].$$

The kernel W_α (see [17]), is even and we have

$$W_\alpha(x,,y,z) = W_\alpha(y,x,z) = W_\alpha(-x,z,y) = W_\alpha(-z,y,-x)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x,y,z)|d\mu_\alpha(z) \leq 4.$$

In the sequel we consider the signed measure $\gamma_{x,y}$, on $\mathbb{R}$, given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x,y,z)d\mu_\alpha(z) & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0. \end{cases} \tag{4}$$

For $x, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, the Dunkl translation operator τ_x is given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z).$$

It was shown in [13] that for $x \in \mathbb{R}$, τ_x is a continuous linear operator from $\mathcal{E}(\mathbb{R})$ into itself and for all $f \in \mathcal{E}(\mathbb{R})$, we have

$$\tau_0(f)(x) = f(x)\,, \quad \tau_x \circ \tau_y = \tau_y \circ \tau_x$$

$$\tau_x(f)(y) = \tau_y(f)(x), \quad \Lambda_\alpha \circ \tau_x = \tau_x \circ \Lambda_\alpha\,, \quad x,\ y \in \mathbb{R}, \tag{5}$$

where $\mathcal{E}(\mathbb{R})$ denotes the space of C^∞-functions on $\mathbb{R}$.
According to [19], the operator τ_x can be extended to $L^p(\mu_\alpha)$, $1 \le p \le +\infty$ and for $f \in L^p(\mu_\alpha)$ we have

$$\|\tau_x(f)\|_{p,\alpha} \le 4\|f\|_{p,\alpha}, \tag{6}$$

and for all $x, \lambda \in \mathbb{R}$, $f \in L^1(\mu_\alpha)$, we have

$$\mathcal{F}_\alpha(\tau_x(f))(\lambda) = E_\alpha(i\lambda x)\mathcal{F}_\alpha(f)(\lambda). \tag{7}$$

Using the change of variable $z = (x,y)_\theta = \sqrt{x^2 + y^2 - 2xy\cos\theta}$, we have also

$$\tau_x(f)(y) = \int_0^\pi \Big[f_e((x,y)_\theta) + \frac{x+y}{(x,y)_\theta} f_o((x,y)_\theta) \Big] d\nu_\alpha(\theta) \tag{8}$$

where

$$f_e((x,y)_\theta) = f((x,y)_\theta) + f(-(x,y)_\theta)\,, \quad f_o((x,y)_\theta) = f((x,y)_\theta) - f(-(x,y)_\theta)$$

and

$$d\nu_\alpha(\theta) = \frac{\Gamma(\alpha+1)}{2\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}(1-\cos\theta)\sin^{2\alpha}\theta d\theta.$$

The Dunkl convolution $f *_\alpha g$, of two continuous functions f and g on $\mathbb{R}$ with compact support, is defined by

$$(f *_\alpha g)(x) = \int_{\mathbb{R}} \tau_x(f)(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

The convolution $*_\alpha$ is associative and commutative (see [17]).
We have the following results (see [18]).

i) Assume that $p, q, r \in [1, +\infty[$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ (the Young condition). Then the map $(f, g) \to f *_\alpha g$ defined on $C_c(\mathbb{R}) \times C_c(\mathbb{R})$, extends to a continuous map from $L^p(\mu_\alpha) \times L^q(\mu_\alpha)$ to $L^r(\mu_\alpha)$ and we have

$$\|f *_\alpha g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}. \tag{9}$$

ii) For all $f \in L^1(\mu_\alpha)$ and $g \in L^2(\mu_\alpha)$, we have

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g) \tag{10}$$

and for $f \in L^1(\mu_\alpha)$, $g \in L^p(\mu_\alpha)$ and $1 \leq p < \infty$, we get

$$\tau_t(f *_\alpha g) = \tau_t(f) *_\alpha g = f *_\alpha \tau_t(g), \quad t \in \mathbb{R}. \tag{11}$$

3. Characterizations of the Besov-Dunkl spaces

In this section, we provide equivalence between different characterizations of the Besov-Dunkl spaces.

Theorem 1. *Let* $1 \leq p < +\infty$, $1 \leq q \leq +\infty$ *and* $\beta > 0$, *then*

$$\mathcal{BD}_{p,q}^{\beta,\alpha} = \mathcal{KD}_{p,q}^{\beta,\alpha}.$$

Proof. For $x > 0$ and $0 < |z| \leq x$, put $\Theta(x, z) = \dfrac{1}{2x^{2\alpha+1}} + \dfrac{sgn(z)}{2|z|^{2\alpha+1}}$.

We start with the proof of the inclusion $\mathcal{BD}_{p,q}^{\beta,\alpha} \subset \mathcal{KD}_{p,q}^{\beta,\alpha}$. For $f \in \mathcal{BD}_{p,q}^{\beta,\alpha}$ and $x > 0$, we take

$$f_1 = \frac{1}{x}\int_{-x}^{x} \Theta(x, z)\, \tau_z(f)\, d\mu_\alpha(z).$$

Using the Minkowski's inequality for integrals and (6), we have

$$\begin{aligned} \|f_1\|_{p,\alpha} &\leq \frac{1}{x}\int_{-x}^{x} |\Theta(x, z)|\, \|\tau_z(f)\|_{p,\alpha}\, d\mu_\alpha(z) \\ &\leq c\,\frac{\|f\|_{p,\alpha}}{x}\int_{-x}^{x} |\Theta(x, z)|\, d\mu_\alpha(z) \ \leq c\|f\|_{p,\alpha}. \end{aligned}$$

By (5) and the generalized Taylor formula with integral remainder (see[14], Theorem 2, p. 349), we get

$$\begin{aligned} \Lambda_\alpha f_1 &= \frac{1}{x}\int_{-x}^{x} \Theta(x, z)\, \tau_z(\Lambda_\alpha f)\, d\mu_\alpha(z) \\ &= \frac{1}{x}(\tau_x(f) - f), \end{aligned}$$

then we obtain,

$$x\|\Lambda_\alpha f_1\|_{p,\alpha} \le c\, w_{p,\alpha}(f,x). \tag{12}$$

On the other hand, put $f_0 = f - 2^{\alpha+2}\Gamma(\alpha+2)f_1$, we can write

$$f_0 = -\frac{2^{\alpha+2}\Gamma(\alpha+2)}{x}\int_{-x}^{x}\Theta(x,z)(\tau_z(f)-f)d\mu_\alpha(z),$$

by the Minkowski's inequality for integrals, we get

$$\begin{aligned}\|f_0\|_{p,\alpha} &\le \frac{c}{x}\int_{-x}^{x}|\Theta(x,z)|\ \|\tau_z(f)-f\|_{p,\alpha}\, d\mu_\alpha(z)\\ &\le c\,\frac{w_{p,\alpha}(f,x)}{x}\int_{-x}^{x}|\Theta(x,z)|\, d\mu_\alpha(z)\\ &\le c\, w_{p,\alpha}(f,x).\end{aligned} \tag{13}$$

Hence by (12) and (13), we deduce that

$$K_{p,\alpha}(f,x) \le c\, w_{p,\alpha}(f,x). \tag{14}$$

Let prove now the inclusion $\mathcal{KD}^{\beta,\alpha}_{p,q} \subset \mathcal{BD}^{\beta,\alpha}_{p,q}$. For $f \in \mathcal{KD}^{\beta,\alpha}_{p,q}$, $x > 0$ and $f_0 \in L^p(\mu_\alpha)$, $f_1 \in \mathcal{D}_{p,\alpha}$ such that $f = f_0 + f_1$, we have by (6)

$$w_{p,\alpha}(f_0,x) \le c\,\|f_0\|_{p,\alpha}\,, \tag{15}$$

on the other hand, using ([14], Theorem 2) we can write for t such that $|t| \le x$

$$\tau_t(f_1) - f_1 = \int_{-|t|}^{|t|}\Theta(t,z)\ \tau_z(\Lambda_\alpha f_1)\, d\mu_\alpha(z),$$

by the Minkowski's inequality for integrals and (6) again, we get

$$\begin{aligned}\|\tau_t(f_1)-f_1\|_{p,\alpha} &\le \int_{-|t|}^{|t|}|\Theta(t,z)|\ \|\tau_z(\Lambda_\alpha f_1)\|_{p,\alpha}\, d\mu_\alpha(z)\\ &\le c\,\|\Lambda_\alpha f_1\|_{p,\alpha}\int_{-|t|}^{|t|}|\Theta(t,z)|\, d\mu_\alpha(z)\\ &\le c\,|t|\ \|\Lambda_\alpha f_1\|_{p,\alpha} \le c\,x\,\|\Lambda_\alpha f_1\|_{p,\alpha}\,,\end{aligned}$$

then we obtain,

$$w_{p,\alpha}(f_1,x) \le c\,x\,\|\Lambda_\alpha f_1\|_{p,\alpha}, \tag{16}$$

since

$$w_{p,\alpha}(f,x) \quad \leq \quad w_{p,\alpha}(f_0,x) + w_{p,\alpha}(f_1,x),$$

by (15) and (16), we deduce that

$$w_{p,\alpha}(f,x) \quad \leq \quad c\, K_{p,\alpha}(f,x). \tag{17}$$

Our theorem is proved. □

Theorem 2. *Let* $1 \leq p \leq 2$, $1 \leq q \leq +\infty$ *and* $\beta > 0$, *then*

$$\mathcal{BD}^{\beta,\alpha}_{p,q} \subset \mathcal{ED}^{\beta,\alpha}_{p,q}.$$

Proof. Let $f \in \mathcal{BD}^{\beta,\alpha}_{p,q}$ and $\lambda,\ x > 0$, by (14) and (17) we have

$$\begin{aligned} w_{p,\alpha}(f,\lambda x) &\leq c\, K_{p,\alpha}(f,\lambda x) \\ &\leq c\, \max\{1,\lambda\} K_{p,\alpha}(f,x) \\ &\leq c\, \max\{1,\lambda\} w_{p,\alpha}(f,x). \end{aligned} \tag{18}$$

Choose $\varphi \in \mathcal{S}_*(\mathbb{R})$ with $\mathrm{supp}\,(\mathcal{F}_\alpha(\varphi)) \subset [-1,1]$ and $\displaystyle\int_{\mathbb{R}} \varphi(x) d\mu_\alpha(x) = 1$. From (10), we get for $t > 0$

$$\mathcal{F}_\alpha(f *_\alpha \varphi_{\frac{1}{t}}) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(\varphi_{\frac{1}{t}})$$

where $\varphi_{\frac{1}{t}}(x) = t^{2(\alpha+1)}\varphi(tx)$, which implies $\mathrm{supp}\,(\mathcal{F}_\alpha(f *_\alpha \varphi_{\frac{1}{t}})) \subset [-t,t]$ and

$$\mathbf{E}_{p,\alpha}(f,t) \leq \|f - f *_\alpha \varphi_{\frac{1}{t}}\|_{p,\alpha}. \tag{19}$$

On the other hand, by the Minkowski's inequality for integrals

$$\begin{aligned} \|f - f *_\alpha \varphi_{\frac{1}{t}}\|_{p,\alpha} &= \Big(\int_{\mathbb{R}} \Big| f(y) - \int_{\mathbb{R}} \varphi_{\frac{1}{t}}(z) \tau_y(f)(z) d\mu_\alpha(z) \Big|^p d\mu_\alpha(y) \Big)^{1/p} \\ &= \Big(\int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \varphi_{\frac{1}{t}}(z) [f(y) - \tau_z(f)(y)] d\mu_\alpha(z) \Big|^p d\mu_\alpha(y) \Big)^{1/p} \\ &\leq \int_{\mathbb{R}} |\varphi_{\frac{1}{t}}(z)|\ \|\tau_z(f) - f\|_{p,\alpha} d\mu_\alpha(z) \\ &\leq \int_{\mathbb{R}} |\varphi_{\frac{1}{t}}(z)|\ w_{p,\alpha}(f,|z|)\ d\mu_\alpha(z), \end{aligned}$$

using (18), we obtain

$$\begin{aligned} \|f - f *_\alpha \varphi_{\frac{1}{t}}\|_{p,\alpha} &\leq c\, w_{p,\alpha}(f, \frac{1}{t}) \int_{\mathbb{R}} |\varphi_{\frac{1}{t}}(z)|\,(1+t|z|\,)d\mu_\alpha(z) \\ &\leq c\, w_{p,\alpha}(f, \frac{1}{t}) \int_{\mathbb{R}} |\varphi(z)|\,(1+|z|\,)d\mu_\alpha(z) \\ &\leq c\, w_{p,\alpha}(f, \frac{1}{t}). \end{aligned} \tag{20}$$

Thus, (19) and (20) imply

$$\begin{aligned} \int_1^{+\infty} \Big(t^\beta \mathbf{E}_{p,\alpha}(f,t)\Big)^q \,\frac{dt}{t} &\leq c \int_0^{+\infty} (t^\beta w_{p,\alpha}(f, \frac{1}{t}))^q \,\frac{dt}{t} \\ &\leq c \int_0^{+\infty} \left(\frac{w_{p,\alpha}(f,t)}{t^\beta}\right)^q \,\frac{dt}{t}\,, \qquad \text{if } q < +\infty \end{aligned}$$

and the same is true for $q = +\infty$.
This completes the proof of the inclusion. □
Now, in order to establish that $\mathcal{BD}^{\beta,\alpha}_{p,q} = \mathcal{ED}^{\beta,\alpha}_{p,q}$ for $1 \leq p \leq 2$, $1 \leq q < +\infty$ and $0 < \beta < 1$, we need to show some useful results.
In the following lemma, we prove a Bernstein-type inequality for the Dunkl translation operators. An analogous result has been proved by [6, 10] for the generalized translation operators associated with the Bessel operator.

Lemma 1. *For $1 \leq p < +\infty$, there exists a constant $c > 0$ such that for $h \in L^p(\mu_\alpha)$ an even differentiable function on $\mathbb{R}$ with $h' \in L^p(\mu_\alpha)$ and $y_1,\ y_2 > 0$, we have*

$$\|\tau_{y_1}(h) - \tau_{y_2}(h)\|_{p,\alpha} \leq c\,|y_1 - y_2|\;\|h'\|_{p,\alpha}.$$

Proof. Using (8) and the fact that h is even, we can assert that

$$\begin{aligned} &\|\tau_{y_1}(h) - \tau_{y_2}(h)\|^p_{p,\alpha} \\ &\quad= \int_{\mathbb{R}} \Big|[\tau_{y_1}(h) - \tau_{y_2}(h)](x)\Big|^p d\mu_\alpha(x) \\ &\quad= \int_{\mathbb{R}} \Big|\int_0^\pi [2h((x,y_1)_\theta) - 2h((x,y_2)_\theta)]\; d\nu_\alpha(\theta)\Big|^p d\mu_\alpha(x) \\ &\quad\leq c \int_{\mathbb{R}} \Big(\int_0^\pi \Big|h((x,y_1)_\theta) - h((x,y_2)_\theta)\Big|^p \; d\nu_\alpha(\theta)\Big) d\mu_\alpha(x) \\ &\quad\leq c \int_{\mathbb{R}} \Big(\int_0^\pi \Big|\int_0^1 \frac{d}{ds}[h((x, y_2 + s(y_1 - y_2))_\theta)]\; ds\Big|^p d\nu_\alpha(\theta)\Big) d\mu_\alpha(x), \end{aligned}$$

since $\dfrac{d}{ds}\Big|(x, y_2 + s(y_1 - y_2))_\theta\Big| \leq |y_1 - y_2|$, then we can write

$$\begin{aligned}
&\|\tau_{y_1}(h) - \tau_{y_2}(h)\|_{p,\alpha}^p \\
&\leq \ c\,|y_1 - y_2|^p \int_{\mathbb{R}} \int_0^\pi \Big| \int_0^1 h'((x, y_2 + s(y_1 - y_2))_\theta)\, ds\Big|^p d\nu_\alpha(\theta) d\mu_\alpha(x), \\
&\leq \ c\,|y_1 - y_2|^p \int_0^1 \int_{\mathbb{R}} \Big(\int_0^\pi \Big|h'((x, y_2 + s(y_1 - y_2))_\theta)\Big|^p \sin^{2\alpha}\theta\, d\theta\Big) d\mu_\alpha(x) ds.
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(\int_0^\pi \Big|h'((x, y_2 + s(y_1 - y_2))_\theta)\Big|^p \sin^{2\alpha}\theta\, d\theta\Big) d\mu_\alpha(x) \\
&\qquad = \int_0^{+\infty} \Big(\int_0^\pi \Big|h'((x, y_2 + s(y_1 - y_2))_\theta)\Big|^p \sin^{2\alpha}\theta\, d\theta\Big) d\mu_\alpha(x) \\
&\qquad + \int_{-\infty}^0 \Big(\int_0^\pi \Big|h'((x, y_2 + s(y_1 - y_2))_\theta)\Big|^p \sin^{2\alpha}\theta\, d\theta\Big) d\mu_\alpha(x).
\end{aligned} \tag{21}$$

By [20], we have for $x \geq 0$,

$$\int_0^\pi \Big|h'((x, y_2 + s(y_1 - y_2))_\theta)\Big|^p \sin^{2\alpha}\theta\, d\theta = c_\alpha\, T_{y_2 + s(y_1 - y_2)}(|h'|^p)(x) \tag{22}$$

where T_y, $y \geq 0$ is the generalized translation operator associated with the Bessel operator and $c_\alpha = \sqrt{\pi}\,\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}$.
On the other hand, by the change of variable $\theta' = \pi - \theta$, we get for $x \leq 0$,

$$\begin{aligned}
&\int_0^\pi \Big|h'((x, y_2 + s(y_1 - y_2))_\theta)\Big|^p \sin^{2\alpha}\theta\, d\theta \\
&\qquad = \int_0^\pi \Big|h'((-x, y_2 + s(y_1 - y_2))_{\theta'})\Big|^p \sin^{2\alpha}\theta'\, d\theta' \\
&\qquad = c_\alpha\, T_{y_2 + s(y_1 - y_2)}(|h'|^p)(-x).
\end{aligned} \tag{23}$$

Then from (21), (22) and (23), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(\int_0^\pi \Big|h'((x, y_2 + s(y_1 - y_2))_\theta)\Big|^p \sin^{2\alpha}\theta\, d\theta\Big) d\mu_\alpha(x) \\
&\qquad = 2c_\alpha \int_0^{+\infty} T_{y_2 + s(y_1 - y_2)}(|h'|^p)(x) d\mu_\alpha(x) \\
&\qquad \leq c \int_0^{+\infty} |h'|^p(x) d\mu_\alpha(x) \leq c\, \|h'\|_{p,\alpha}^p.
\end{aligned}$$

Hence, we deduce

$$\|\tau_{y_1}(h) - \tau_{y_2}(h)\|_{p,\alpha} \leq c\,|y_1 - y_2|\; \|h'\|_{p,\alpha},$$

which proves the result. $\square$

Lemma 2. *For $1 \leq p \leq 2$, there exists a constant $c > 0$ such that for any $x > 0$, any function $g \in L^p(\mu_\alpha)$ with supp $(\mathcal{F}_\alpha(g)) \subset [-x,\, x]$ and $y_1, y_2 > 0$, we have*

$$\|\tau_{y_1}(g) - \tau_{y_2}(g)\|_{p,\alpha} \leq c\, x\,|y_1 - y_2|\; \|g\|_{p,\alpha}.$$

Proof. Let $g \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}(\mathcal{F}_\alpha(g)) \subset [-x,\, x]$. Choose $\varphi \in \mathcal{D}_*(\mathbb{R})$ such that $\varphi(t) = 1$ if $|t| \leq 1$ and $\varphi(t) = 0$ if $|t| \geq 2$. Then by the inversion formula (2), we have $\varphi = \mathcal{F}_\alpha(h)$ for some $h \in \mathcal{S}_*(\mathbb{R})$. Put $h_x(y) = x^{2(\alpha+1)} h(xy)$ for $y \in \mathbb{R}$, then $\mathcal{F}_\alpha(h_x)(y) = \varphi(\frac{y}{x}) = 1$ for $|y| \leq x$. Note that $\operatorname{supp}(\mathcal{F}_\alpha(g)) \subset [-x,\, x]$, then using (1), (7) and (10), we can write

$$\mathcal{F}_\alpha(\tau_{y_1}(g) - \tau_{y_2}(g)) = \mathcal{F}_\alpha(h_x *_\alpha (\tau_{y_1}(g) - \tau_{y_2}(g))),$$

by (2) and (9), we obtain

$$\begin{aligned} \tau_{y_1}(g) - \tau_{y_2}(g) &= h_x *_\alpha (\tau_{y_1}(g) - \tau_{y_2}(g)) \\ &= (\tau_{y_1}(h_x) - \tau_{y_2}(h_x)) *_\alpha g. \end{aligned}$$

The change of variable $t' = xt$ in (3) gives

$$W_\alpha(xy, xz, t')\, x^{2(\alpha+1)} = W_\alpha(y, z, t),$$

then from (4), we get

$$d\gamma_{xy,xz}(t') = d\gamma_{y,z}(t) \;\text{ and }\; \tau_y(h_x)(z) = x^{2(\alpha+1)}\tau_{xy}(h)(xz).$$

Therefore, using the lemma 1 , we have

$$\begin{aligned} \|\tau_{y_1}(g) - \tau_{y_2}(g)\|_{p,\alpha} &\leq 4\,\|\tau_{y_1}(h_x) - \tau_{y_2}(h_x)\|_{1,\alpha}\; \|g\|_{p,\alpha} \\ &= 4\,\|\tau_{xy_1}(h) - \tau_{xy_2}(h)\|_{1,\alpha}\; \|g\|_{p,\alpha} \\ &\leq c\,x\,|y_1 - y_2|\; \|h'\|_{p,\alpha}\|g\|_{p,\alpha} \\ &\leq c\,x\,|y_1 - y_2|\; \|g\|_{p,\alpha}. \end{aligned}$$

Since $\mathcal{S}(\mathbb{R})$ is a dense subset of $L^p(\mu_\alpha)$ for $1 \leq p < +\infty$ and by (6), we obtain the result. $\square$

Theorem 3. *Let $1 \leq p \leq 2$, $1 \leq q < +\infty$ and $0 < \beta < 1$, then*

$$\mathcal{ED}^{\beta,\alpha}_{p,q} = \mathcal{BD}^{\beta,\alpha}_{p,q}.$$

Proof. We have only to show that $\mathcal{ED}^{\beta,\alpha}_{p,q} \subset \mathcal{BD}^{\beta,\alpha}_{p,q}$. Assume $f \in \mathcal{ED}^{\beta,\alpha}_{p,q}$, we can consider $f \neq 0$ a.e., then we get

$$\begin{aligned} \Big(\int_0^1 (t^{-\beta} w_{p,\alpha}(f,t))^q \,\frac{dt}{t}\Big)^{\frac{1}{q}} &= \Big(\sum_{n=0}^{+\infty} \int_{2^{-n-1}}^{2^{-n}} (t^{-\beta} w_{p,\alpha}(f,t))^q \,\frac{dt}{t}\Big)^{\frac{1}{q}} \\ &\leq 2^{\beta}\Big(\sum_{n=0}^{+\infty} (2^{n\beta} w_{p,\alpha}(f,2^{-n}))^q\Big)^{\frac{1}{q}} \\ &= 2^{\beta} \sum_{n=0}^{+\infty} \lambda_n 2^{n\beta} w_{p,\alpha}(f,2^{-n}), \end{aligned}$$

where $\lambda_n = \dfrac{(2^{n\beta} w_{p,\alpha}(f,2^{-n}))^{\frac{q}{q'}}}{\Big(\sum_{n=0}^{+\infty}(2^{n\beta} w_{p,\alpha}(f,2^{-n}))^q\Big)^{\frac{1}{q'}}}$ with q' the conjugate of q.

By reasoning as in the proof on ([16], Proposition 3.1, p. 88) and using the lemma 2, we have for $0 < \beta < 1$,

$$\Big(\int_0^1 (t^{-\beta} w_{p,\alpha}(f,t))^q \,\frac{dt}{t}\Big)^{\frac{1}{q}} \leq 2^{\beta} c\Big(\|f\|_p + (\sum_{m=1}^{+\infty} (2^{m\beta}\mathbf{E}_{p,\alpha}(f,2^{m-1}))^q)^{\frac{1}{q}}\Big)$$

Since $\mathbf{E}_{p,\alpha}(f,t)$ is decreasing in t and by (19),

$$\begin{aligned} \Big(\sum_{m=1}^{+\infty} (2^{m\beta}\mathbf{E}_{p,\alpha}(f,2^{m-1}))^q\Big)^{\frac{1}{q}} &= 2^{\beta}\mathbf{E}_{p,\alpha}(f,1) + \Big(\sum_{m=2}^{+\infty} (2^{m\beta}\mathbf{E}_{p,\alpha}(f,2^{m-1}))^q\Big)^{\frac{1}{q}} \\ &\leq c\Big(\|f\|_p + \Big(\int_1^{+\infty} \Big(t^{\beta}\mathbf{E}_{p,\alpha}(f,t)\Big)^q \,\frac{dt}{t}\Big)^{\frac{1}{q}}\Big). \end{aligned}$$

The result of the two inequalities above is

$$\Big(\int_0^1 (t^{-\beta} w_{p,\alpha}(f,t))^q \,\frac{dt}{t}\Big)^{\frac{1}{q}} \leq c\Big(\|f\|_p + \Big(\int_1^{+\infty} \Big(t^{\beta}\mathbf{E}_{p,\alpha}(f,t)\Big)^q \,\frac{dt}{t}\Big)^{\frac{1}{q}}\Big).$$

On the other hand, we easily obtain,

$$\Big(\int_1^{+\infty} (t^{-\beta} w_{p,\alpha}(f,t))^q \,\frac{dt}{t}\Big)^{\frac{1}{q}} \leq c\,\|f\|_p\Big(\int_1^{+\infty} t^{-\beta q-1}\,dt\Big)^{\frac{1}{q}} \leq c\,\|f\|_p.$$

Hence, we conclude that

$$\Big(\int_0^{+\infty}(t^{-\beta}w_{p,\alpha}(f,t))^q\,\frac{dt}{t}\Big)^{\frac{1}{q}}\le c\Big(\|f\|_p+\Big(\int_1^{+\infty}\Big(t^{\beta}\mathbf{E}_{p,\alpha}(f,t)\Big)^q\;\frac{dt}{t}\Big)^{\frac{1}{q}}\Big).$$

This completes the proof. □

References

1. C. Abdelkefi and M. Sifi, On the uniform convergence of partial Dunkl integrals in Besov-Dunkl spaces. Fractional Calculus and Applied Analysis Vol. 9, N. 1 (2006), 43-56.

2. C. Abdelkefi and M.Sifi, Further results of integrability for the Dunkl transform. Communication in Mathematical Analysis Vol. 2, N.1 (2007), 29-36.

3. C. Abdelkefi and M. Sifi, Characterization of Besov spaces for the Dunkl operator on the real line. Submitted to Journal of Inequalities in Pure and Applied Mathematics.

4. J. L. Ansorena and O. Blasco, Characterization of weighted Besov spaces, Math. Nachr. 171 (1995), 5-17.

5. O. V. Besov, On a family of function spaces in connection with embedding and extensions, Trudy Mat. Inst. Steklov 60 (1961), 42-81.

6. J. J. Betancor, J. M. Méndez and L. Rodríguez-Mesa, Espacios de Besov Asociados a la Transformación de Hankel, Servicio de Publicaciones, Universidad de la Rioja, Logroño, Spain, 2001.

7. M. F. E de Jeu, The Dunkl transform , Inv. Math. 113 (1993) 147-162.

8. C.F. Dunkl, Differential-difference operators associated to reflection groups, Trans.Amer. Math. Soc. 311, No1, (1989), 167-183.

9. C.F. Dunkl, Hankel transforms associated to finite reflection groups in : Proc. of special session on hypergeometric functions on domains of positivity, Jack polynomials and applications. proceedings, Tampa 1991, Contemp. Math. 138 (1992), 123-138.

10. J. Gosselin and K. Stempak, A weak type estimate for Fourier-Bessel multipliers, Proc. Amer. Math. Soc. 106 (1989), 655-662.

11. L. Kamoun, Besov-type spaces for the Dunkl operator on the real line. Journal of Computational and Applied Mathematics 199, N. 1 (2007), 56-67.

12. J. Löfström and J. Peetre, Approximation theorems connected with generalized translations, Math. Ann. 181 (1969), 255-268.

13. M.A. Mourou, Transmutation operators associated with a Dunkl-type differential-difference operator on the real line and certain of their applications, Integral Transforms Spec. Funct. 12, No 1 (2001), 77-88.

14. M.A. Mourou, Taylor series associated with a differential-difference operator on the real line. Journal of Computational and Applied Mathematics, Vol. 153 (2003), 343-354.

15. J. Peetre, New thoughts on Besov spaces, Duke Univ. Math. Series, Durham, NC, 1976.

16. A. Pelczyński and M. Wojciechowski, Molecular decompositions and embedding theorems for vector-valued Sobolev spaces with gradient norm, Studia Math. 107 (1993), 61-100.

17. M. Rösler, Bessel-Type signed hypergroup on $\mathbb{R}$, in Probability measure on groups and related structures, Proc. Conf. Oberwolfach, (1994), H. Heyer and A. Mukherjea (Eds) World Scientific Publ, 1995, 292-304.

18. F. Soltani, L^p-Fourier multipliers for the Dunkl operator on the real line, J. Funct. Analysis, 209 (2004), 16-35.

19. S. Thangavelyu and Y. Xu, Convolution operator and maximal function for Dunkl transform. J. Anal. Math., vol. 97, pp. 25-55 (2005).

20. K. Trimèche, Generalized Harmonic Analysis and Wavelets Packets. Gordon and Breach Science Publichers 2001 OPA.

21. H. Triebel, Theory of function spaces, Monographs in Math., Vol. 78, Birkhäuser, Verlag, Basel, 1983.

22. G.N. Watson, A treatise on the theory of Bessel functions, Camb. Univ. Press, Cambridge 1966.